THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 7

§47) 1) a) Note that on the contour C, by triangle inequality, we have

$$|z+4| \leq |z|+4 = 2+4 = 6 \text{ and}$$
$$|z^3-1| \geq |z|^3 - 1 = 8-1 = 7$$

Thus, we have

$$\left|\int_C \frac{z+4}{z^3-1}dz\right| \leq \frac{6}{7} \times \text{length of the contour } C = \frac{6\pi}{7}.$$

§47) 4) Note that on the contour C, by triangle inequality, we have

$$|2z^2-1| \leq 2|z|^2+1 = 2R^2+1 \text{ and}$$
$$|z^4+5z^2+4| = |z^2+1||z^2+4| \geq (|z|^2-1)(|z|^2-4) = (R^2-1)(R^2-4)$$

Thus, we have

$$\left|\int_C \frac{2z^2-1}{z^4+5z^2+4}dz\right| \leq \frac{2R^2+1}{(R^2-1)(R^2-4)} \times \text{length of the contour } C$$
$$= \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}.$$

In particular, since $\lim_{R\to\infty} \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)} = \lim_{R\to\infty} \frac{1}{R}\frac{\pi(2+\frac{1}{R^2})}{(1-\frac{1}{R^2})(1-\frac{4}{R^2})} = 0$, the value of the integral tends to zero as R tends to infinity.

§47) 5) Note that on the contour $C : |z| = R$, by triangle inequality, we have

$$|\operatorname{Log} z| = |\ln|z| + i\operatorname{Arg} z| \leq |\ln|z|| + i|\operatorname{Arg} z| \leq \ln R + \pi \text{ and } |z^2| = R^2$$

Thus, we have

$$\left|\int_C \frac{\operatorname{Log} z}{z^2}dz\right| \leq \frac{\pi+\ln R}{R^2} \times \text{length of the contour } C$$
$$= \frac{\pi+\ln R}{R^2} \times (2\pi R)$$
$$= 2\pi\left(\frac{\pi+\ln R}{R}\right).$$

Since $\lim_{R\to\infty} 2\pi\left(\frac{\pi+\ln R}{R}\right) = \lim_{R\to\infty} 2\pi\left(\frac{\frac{1}{R}}{1}\right) = 0$, the value of the integral tends to zero as R tends to infinity.

Remark: As pointed out by some of you, this question requires you to show the *strict* inequality. To do so, let C_1, C_2 and C_3 be the positively oriented contours defined by

$C_1 = \{Re^{i\theta} \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$, $C_2 = \{Re^{i\theta} \mid \frac{\pi}{2} \le \theta \le \pi\}$ and $C_3 = \{Re^{i\theta} \mid -\pi \le \theta \le \frac{-\pi}{2}\}$.

Then $C = C_1 \cup C_2 \cup C_3$. Note that on C_1, we have the *strict* inequality, i.e.

$$|\operatorname{Log} z| = |\ln|z| + i \operatorname{Arg} z| \le |\ln|z|| + i|\operatorname{Arg} z| < \ln R + \pi$$

Since $\left|\int_C \frac{\operatorname{Log} z}{z^2} dz\right| \le \sum_{i=1}^{3} \left|\int_{C_i} \frac{\operatorname{Log} z}{z^2} dz\right|$ by triangle inequality, we get the desired result.

§49) 2) b) $\displaystyle\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz = 2\left[\sin\left(\frac{z}{2}\right)\right]_0^{\pi+2i} = 2\sin\left(\frac{\pi}{2}+i\right) = 2 \times \frac{e^{i(\frac{\pi}{2}+i)} - e^{-i(\frac{\pi}{2}+i)}}{2i} = e + \frac{1}{e}.$

c) $\displaystyle\int_1^3 (z-2)^3 dz = \left[\frac{(z-2)^4}{4}\right]_1^3 = \frac{1}{4} - \frac{1}{4} = 0.$

§49) 5) Let $\log z$ be the logarithmic function with domain $|z| > 0$ and $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$. Then

$$\int_{-1}^{1} z^i dz = \left[\frac{z^{i+1}}{i+1}\right]_{-1}^{1} = \frac{(1)^{i+1}}{i+1} - \frac{(-1)^{i+1}}{i+1} = \frac{1}{1+i}[\exp(i\log(1)) - \exp(i\log(-1)] = \frac{1-i}{2}(1+e^{-\pi})$$

§53) 1) c) Note that $z^2 + 2z + 2 = 0 \iff z = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = -1 \pm i$. In particular, that means the function $f(z)$ is analytic on and inside the contour $|z| = 1$. By Cauchy-Goursat theorem, we have $\int_C f(z)dz = 0$.

f) Note that $\operatorname{Log}(z+2)$ is not analytic if and only if $(z+2) = -r$ for some $r \ge 0$, i.e. $z = -2-r$. In particular, that means the function $f(z)$ is analytic on and inside the contour $|z| = 1$. By Cauchy-Goursat theorem, we have $\int_C f(z)dz = 0$.

§53) 2) b) Note that $f(z)$ is not analytic if and only if $\sin(z/2) = 0$, i.e. $z = 2n\pi$ for some $n \in \mathbb{Z}$. In particular, those singularities do not lie on the region bounded by C_1 and C_2.
Therefore, we have $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

§53) 3) Let $R > 0$ be a positive real number such that the circle $\Gamma : |z - (2+i)| = R$ lie completely inside the rectangle C. Since the function $(z-2-i)^{n-1}$ is analytic on the region bounded by the circle and the rectangle, we have

$$\int_C (z-2-i)^{n-1} dz = \int_\Gamma (z-2-i)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots \\ 2\pi i & \text{when } n = 0. \end{cases}$$